Remembering the "Little Prince of Mathematics"

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1 The Little Prince

ANTOINE DE SAINT-EXUPÉRY, born in France in 1900, was not a professional writer, but an airplane pilot in the French mail service. During the second World War, he served a French air squadron in Northern Africa. On July 31, 1944, he set out from Borgo, in Corsica, to overfly occupied France, and never returned.

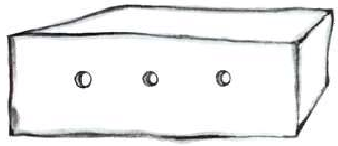
In 1943, SAINT-EXUPÉRY had published his last book *The Little Prince* (*Le Petit Prince*).

This delightful essay tells the story of the author, forced to land in the desert because of a mechanical failure. When he tries to repair the airplane, a little boy comes from nowhere and asks him :

> please draw me a sheep.

SAINT-EXUPÉRY is a gifted man, except for drawing. After several unsuccessful trials, all rejected by the little boy for one or another reason, he finally draws a box, telling :

the sheep is inside.



The little boy, who appears to be a little prince coming from a small planet, is delighted. He will see inside the box a number of features of the sheep that SAINT-EXUPÉRY never thought about !

In 1989, a young Portuguese mathematician, sent by LUIS SANCHEZ, arrived at Louvain-la-Neuve to prepare a PhD in our Department, sponsored by the *Instituto Nacional de Investigação Cientifica of Portugal.* His name was MIGUEL RAMOS, he was 26 years old and looked much younger, as he always will do. Very soon, MIGUEL used to knock at my office's door, to chat, or, more often, to ask me some questions. Those questions were always amazingly sharp and fundamental, going to the heart of the problem, even when they looked somewhat naive.

I was familiar with SAINT-EXUPÉRY's *Little Prince* for having read the book several times, and even performed in high school time the role of the astronomer in a play based upon the novel. Almost immediately, I identified MIGUEL (without telling him of course) with the "Little Prince". Besides his kind and young appearance, his sweet eyes and smile, the main reason was that the best answer I could in general give to his questions was like drawing a box. In many occasions, MIGUEL saw in the box things which had escaped me. He remains in my memory the 'Little Prince of Mathematics".

But no story of a prince is complete without a princess. And it happened that I had no more insight in this affair than in answering MIGUEL's questions. Not long before the end of MIGUEL' stay in Louvain-la-Neuve, the chairman of the department, PAUL HENRARD in this time, entered my office with an angry face and told me :

> Jean, you should watch on your PhD students in a more efficient way !.

I could hardly imagine that MIGUEL could be involved in any criminal or even rough behavior. The "crime", so to say, was that MIGUEL had found his princess among the secretaries of our Departement, and was decided to bring her with him to Portugal. Needless to say that, with his good taste, Miguel had chosen our best secretary, Béatrice. This was the main reason of the chairman's disappointment and faint anger. I must confess that I had neither realized, nor even suspected this love affair. Were the lovers extremely cautious or was I exceptionally blind in this domain ? I bet my wife would chose the second solution without any hesitation.

2 Before the thesis

The Little Prince of Mathematics had a mathematical life before his PhD thesis. In papers [1, 3], he gave extensions of some **multiplicity results of Ambrosetti-Prodi type** obtained by Fabry, Nkashama and myself, and S.H. Ding and myself, for periodic solutions of some differential equations depending upon a parameter and having coercive nonlinearities. The problem consisted in studying the multiplicity of the solutions of the periodic problem

$$\begin{array}{rcl} \pm u^{(m)} + g(t,u) & = & s, \\ u^{(j)}(0) = u^{(j)}(2\pi) & & (j = 0, 1, \ldots, m-1) \end{array} \qquad (1)$$

in terms of the parameter s for some classes of functions g such that

$$\lim_{|u| \to \infty} g(t,u) = +\infty$$

uniformly in t.

Miguel and Luis introduced several new classes of nonlinearities, such that there exists $s_0 \leq s_1$ with the property that *problem (1) has no solution, at least one or at least two solutions when* $s < s_0$, $s = s_1$, $s > s_1$. Cases where $s_0 = s_1$ were discussed as well. The results included situations where g is singular at 0, with the coercivity replaced by the variant

$$\lim_{u \to 0+} g(t,u) = \lim_{u \to +\infty} g(t,u) = +\infty.$$

The proofs used topological degree theory.

Another contribution of the pre-thesis period, which shows Miguel's early interest for **variational methods** as well, was a joint paper with Luis Sanchez [2] on the **Dirichlet problem at resonance**

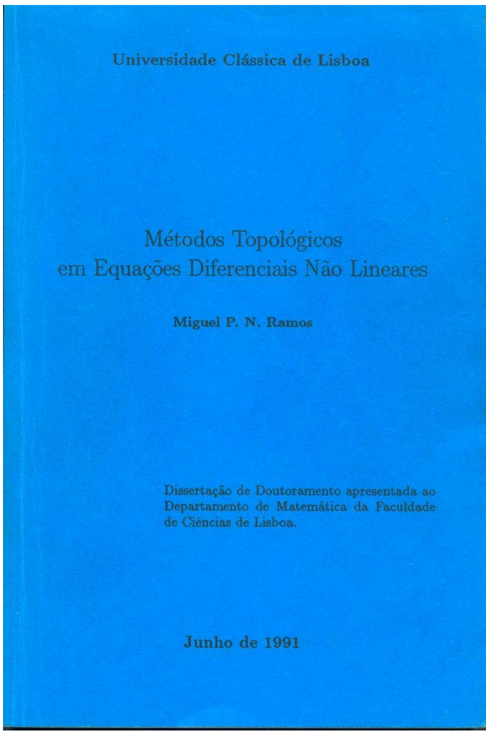
$$\Delta u + \lambda_1 u + f(x,u) = h(x) \;\text{ in }\; \Omega, \quad u = 0 \;\text{ on }\; \partial\Omega$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω and h is orthogonal to the first eigenfunction. The assumptions are related to the sign condition $\pm uf(t,u) \geq 0$ and the Landesman-Lazer conditions and the direct method of the calculus of variations and RABINOWITZ' saddle point are used.

3 The thesis

Miguel's thesis is entitled *Metodos topológicos em Equações Diferenciais Não Lineares (Topological methods in nonlinear differential equations).* Its table of contents goes as follows :

1. Relative category and critical point theory
2. Periodic solutions for weak singular systems
3. Subharmonic oscillations for second order equations
4. Existence and multiplicity near eigenvalues
5. Asymmetric nonlinearities with positive forcing



Chapter 1 clearly describes a variational method, and introduces an extension of relative category, called **limit relative category** which allows the treatment of indefinite functionals. It is motivated by earlier results of GILLES FOURNIER and MICHEL WILLEM. Abstract existence theorems for critical points are given, together with applications to **T-periodic solutions of Hamiltonian systems of the form**

$$J\dot{z} + \nabla H(t, z) = h(t)$$

with $\nabla H(t, z)$ asymptotically linear at infinity, say

$$\nabla H(t, z) = A_\infty(t)z$$

the T-periodic solutions of

$$J\dot{z} + A_\infty(t)z = 0$$

form a k-dimensional space spanned by $\{v_1, \ldots, v_n\}$ with the v_j constant, and

$$H(\cdot, z + v_j) = H(\cdot, z), \quad \int_0^T (h(t), v_j)\, dt = 0,$$

$$(j = 1, \ldots, k).$$

They provide an extension of results of CONLEY-ZEHNDER, K.C. CHANG, myself and FONDA and myself, to situations where H needs only to be of class C^1, and not of class C^2 as requested by the LYAPUNOV-SCHMIDT's reduction argument used by the other authors. Chapter 1 is the basis of a paper with GILLES FOURNIER, DANIELA LUPO and MICHEL WILLEM [6],

Chapter 2 considers periodic solutions of autonomous systems of the form

$$\ddot{u} + \nabla F(u) = 0$$

with weak singular potential

$$\frac{a}{|u|^\alpha} \leq -F(u) \leq \frac{b}{|u|^\alpha}$$

$$-\alpha_1 F(u) \leq \nabla F(u) \cdot u \leq -\alpha_2 F(u)$$

for some $b \geq a > 0$ and $1 > \alpha_2 \geq \alpha \geq \alpha_1 > 0$. The authors use the saddle point theorem. Both solutions for fixed period and for fixed energy are considered. Chapter 2 is puplished as a joint paper with SUSANNATERRACINI [12].

Chapter 3 proves the existence of **subharmonic solutions** of second order systems in $\mathbb{R}^N$, It is the basis of two papers with ALESSANDRO FONDA [11] and ALESSANDRO FONDA and MICHEL WILLEM [9]. The first one uses the saddle point lemma to prove the *existence of kT-periodic solutions which are not T-periodic for equation*

$$\ddot{u} + g(t, u) = e(t)$$

when G is subquadratic at infinity, sgn $ug(t, u)$ *is bounded below and e satisfies the Landesman-Lazer conditions*. Similar conclusions are also obtained in the case of asymmetric nonlinearities.

The second one completes the conclusions of the mountain pass and saddle point lemmas to the corresponding functional

$$\varphi_k(u) = \int_0^{kT} \left(\frac{|\dot{u}(t)|^2}{2} - G(t, u(t)) \right) dt$$

by using Morse theory and Bott iteration formula. They obtain the *existence of kT-periodic solutions (not T-periodic) for all sufficiently large k for the N-dimensional system*

$$\ddot{u} + \nabla_u G(t, u) = 0. \tag{2}$$

when $G(t,\cdot)$ is convex, satisfies the AHMAD-LAZER-PAUL *condition*

$$\lim_{|u|\to\infty} \int_0^T G(t, u)\, dt = +\infty,$$

$\nabla_u G$ is bounded, and no T-periodic solution $u(t)$ of (2) such that $\nabla_u G(t, u(t)) \equiv 0$. For $N = 1$ the same conclusion is true if the Ahmad-Lazer-Paul condition is replaced by a subquadratic condition for G at infinity and a nonresonance condition for g at the first positive eigenvalue. Conditions for subharmonic solutions are also obtained in the Ambrosetti-Prodi type situation for equation

$$\ddot{u} + g(t, u) = s.$$

Chapter 4 finds conditions for the existence and multiplicity of solutions of **semilinear elliptic Dirichlet problems** of the form

$$\Delta u + \lambda u + g(x, u) = h(x) \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega$$

both in **non-resonant and resonant cases**. The saddle point theorem is again used. Sharper results were obtained for the **Neumann boundary value problem** associated to ordinary differential equations of the form

$$\ddot{u} + a(t)g(u + v(t)) = h(t)$$

when a may change sign but has a negative mean value. One also finds extensions of **multiplicity results near an eigenvalue** λ_k with eigenfunction $\varphi_k(t)$ under a Landesman-Lazer condition for Dirichlet problems associated to

$$\ddot{u} + \lambda u + g(u) = s\varphi_k(t).$$

The approach is a combination of shooting method and fixed point theory. Part of it is published in the form of a joint paper with DANIELA LUPO ([7, 5]).

Chapter 5 is dedicated to the existence problem of **solutions of constant sign for Neumann problems** associated to differential equations with **jumping (asymmetric) nonlinearities**

$$\ddot{u} + \mu(t)u^+ - \nu(t)u^- = p(t, u).$$

It is published in the form of three joint papers with PATRICK HABETS and LUIS SANCHEZ [4, 8, 10].

After this description, the **title** *Topological methods in nonlinear differential equations* looks to me a little strange to-day, for a thesis whose four chapters over five develop and/or apply critical point theory. Topological techniques are only used in a part of Chapter 4 and (through Schauder's fixed point theorem only) in Chapter 5. I do not remember if there were reasons for this; maybe a constraint associated to the project of the INIC grant, or Miguel's intention to insist on the fact that topological tools are requested in critical point theory as soon as you leave the direct method of calculus of variations.

One should notice the remarkable **variety** of treated problems in this thesis, covering a two years period only, and reflecting also various **collaborations** Miguel had developed during the preparation. Apparently, when questioned by the little prince, they were able to do better than drawing a box.

I still have a vivid memory of the **defense of the thesis**, which took place in Lisbon at the very beginning of December 1991. According to the rules of the time, two of the members of the jury – which happened to be Luis Sanchez and myself – had to make comments or critics during less than half an hour each, to which the candidate had to answer. Needless to say that Miguel was quied and charming as usual, answering all questions with ease. I do not know if the tradition still exists, but the total time of the defense, including the comments of the jury and answers of the candidate, was measured by an enormous egg-timer. The one which determined Miguel's lifetime was definitely undersized.

All Miguel's papers following those involved in the material of the thesis, will deal with **critical point theory** and its applications. He had extremely rapidly evaluated the potential of this rich and versatile approach, and remained faithful to it till the end of his life.

4 Min-max critical point theorems

Miguel not only made new striking applications of the min-max critical point theorems to ordinary and partial differential equations, but was constantly concerned by their **generalization** and their **unification**. This was a recurrent topics when the Little Prince knocked at my door and asked me to draw a sheep. His arrival to Louvain-la-Neuve coincided almost with the publication of the joint monograph *Critical Point Theory and Hamiltonian Systems* with Michel Willem. It was too late for him to help us in making it better and, for this reason he does not appear in the list of persons quoted in the acknowledgements. But, in a premonitory way maybe, the authors were

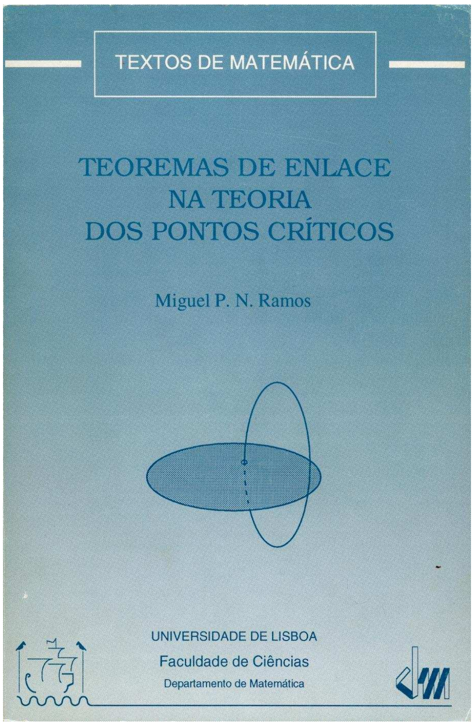
> very grateful to Béatrice Huberty for her accurate and superb typing of the manuscript.

In a way one can say that Béatrice knew more of critical point theory than Miguel when he arrived to Louvain-la-Neuve.

Miguel's reflexions on critical point theory led him between 1993 and 1995 to several publications concerning elegant **unifications** and striking **extensions of various mini-max theorems**. A source of inspiration and motivation was undoubtly the *seminars on nonlinear differential equations and variational methods* given at Lisbon in the academic years 1991/92 and 1992/93, and to which Luis Sanchez, Maria de Rosario Grossinho, Carlota Rebelo, Ma To Fu, Ana Rute Domingos and Julia Matos actively participated.

The first result was a **monograph** (in Portuguese) entitled *Linking theorems in critical point theory*, and published in 1993 as volume 2 of the series *Textos de Matemática* of the Department of Mathematics of Lisbon University, following volume 1 *Methods of the theory of critical points* published the same year by Luis Sanchez. The content of Miguel's volume goes as follows.

1. homotopies between level sets
2. homotopy and compactness
3. minimization
4. linked sets
5. linked sets by homotopy
6. mountain pass theorem
7. saddle point theorem
8. linked spheres theorem
9. multiplicity theorems
10. Morse lemma
11. Sard-Smale theorem
12. critical groups



As written on the back page

> The results are motivated by geometrical situations (called linking) which occur in the variational treatment of some nonlinear differential equations.

Saint-Exupéry's Little Prince was also facing severe linking problems on his small planet !

One surely can regret that those volumes were never translated in English. Even the most standard tools and results of critical point theory are often presented with a touch of originality, and new techniques and results were given for the first time, before being published as separated articles in some papers, alone [16] or in collaboration with Carlotta Rebello [13] and Luis Sanchez [15].

In [13], Ramos and Rebelo intended

> to give a unified presentation of some results of critical point theory which appeared or have been used under a number of variants in the literature in recent years.

The deformation theorems were presented through the use of the concept of **homotopical linking** introduced by Benci-Rabinowitz and Silva. This concept was also be used with Sanchez in [15], where **Morse index estimates** for the critical point are obtained in situations covered by this class of linking. Applications were given, among others, to subharmonic solutions of second order differential equations. In [16], a **three critical point theorem** is stated and proved, motivated by semilinear Dirichlet problems with jumping nonlinearities.

5 Variational approach to multiplicity results near resonance

With this strong variational background, it is not suprising that Miguel came back to the problem of **multiplicity of solutions near resonance** considered earlier, for some ordinary differential equations, with Daniela Lupo, using degree theory. In collaboration with Luis Sanchez, Miguel considered the case of the Dirichlet problem

$$\pm(\Delta u + \lambda u) + f(x,u) = \widetilde{h}(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

near the first eigenvalue λ_1, combining several variational techniques [18] (see also [19]. *Under some general versions of the Landesman-Lazer condition at $\lambda = \lambda_1$, the existence of at least three solutions was proved either for $\lambda < \lambda_1$ and close, or $\lambda > \lambda_1$ and close.* Those results have been developed and extended by Ma To Fu, Sanchez and others to other classes of equations.

Those results were motivated by earlier ones of Klaus Schmitt and myself obtained by a combination of bifurcation at infinity and topological degree. When I tried not long ago, with Cristian Bereanu and Petru Jebelean, to obtain similar results for radial solutions of quasilinear Neumann problems of the extrinsic mean curvature type

$$-\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = f(x,u) \text{ in } B_1(0),$$
$$\partial_\nu u = 0 \text{ on } \partial B_1(0)$$

around the "eigenvalue" 0, there was no bifurcation from infinity available. But Ramos-Sanchez' approach could perfectly be adapted to this new setting and gave us the expected results.

6 Dirichlet problems with asymmetric nonlinearities

Motivated by the seminal work of Ambrosetti-Prodi, of Fučik, of Dancer and of Lazer-McKenna, **Dirichlet problems with asymmetric nonlinearities**

$$\begin{aligned} \Delta u + \alpha u^+ - \beta u^- + g(x,u) &= h(x) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{3}$$

with $\lim_{|u|\to\infty} \frac{g(x,u)}{u} = 0$, have been an important source of inspiration for Miguel.

With Ana Rute Domingos [14, 22] or alone [16], he has found new *existence and multiplicity theorems for problems (3) with*

$$h(x) = s\varphi(x) + \widetilde{h}(x)$$

with φ an eigenfunction associated to the first eigenvalue λ_1, $\widetilde{h}$ orthogonal to φ, and s sufficiently large, when (α, β) belong to suitable subsets of the complement of the Fučik spectrum in $\mathbb{R}^2$.

7 Dirichlet problems with sign changing nonlinearities

From 1997, MIGUEL has contributed to no less than eight papers, with seven coauthors, dealing with the difficult problem of **sign changing nonlinearities** of the type

$$-\Delta u = \mu u + a(x)g(u) \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega,$$

[20, 21, 25] and their extension to biharmonic operator [24, 28], to the whole space [26, 29] and to the biharmonic operator on the whole space [31].

In the seminal paper with SUSANNA TERRACCINI and Christophe Troestler [21] (announced in [20]), the way is paved by showing essentially that *if a changes sign in Ω, satisfies non-degeneracy conditions where a vanishes or on $\partial\Omega$,*

$$g(0) = g'(0) = 0, \lim_{|u|\to\infty} \frac{g'(u)}{(p-1)|u|^{p-2}} = l > 0$$

for some $p \in (2, 2N/(N-2))$, and μ lies between two consecutive eigenvalues λ_k and λ_{k+1}, then a nontrivial solution exists. In the other papers, those conditions are adapted to the specificities of the new situations.

8 Perturbation from symmetry for superlinear problems

It is not surprising that MIGUEL's taste for sharp inequalities and technically elaborate critical point theory has led him to contribute to the **multiplicity of solutions of Dirichlet problems** of the type

$$-\Delta u = g(x,u) + f(x,u) \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega$$

when $g(x,\cdot)$ is **odd, superlinear** in a suitable way and f has a **lower order**. Initiated, for sufficiently small f by MARK A. KRASNOSEL'SKII (that MIGUEL had met in Louvain-la-Neuve and for whom he had a great admiration) in the ninenteen fifties, the problem had seen pioneering independent work by STRUWE and BAHRI-BERESTICKI, improved by Bahri-Lions and many others.

With HOSSEINTEHRANI [37], MIGUEL has proved the *existence of infinitely many solutions for problems with perturbed changing sign superlinear nonlinearities*

$$-\Delta u = \lambda u + a(x)g(u) = f(x) \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega$$

where g is odd, satisfies suitable superlinearity conditions, a changes signs and verifies the non-degeneracy conditions previously mentioned.

With HUGO TAVARES and WEN MING ZOU [40], MIGUEL has proved the *existence of infinitly many sign changing solutions for superlinear problems of the form*

$$-\Delta u = g(x,u) + f(x,u) \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega$$

when $g(x,\cdot)$ is odd, satisfies suitable superlinearity conditions, f satisfies some growth restrictions and both $f(x,\cdot)$ and $g(x,\cdot)$ are superlinear at 0 (which excludes the case where $f(x,u) = f(x)$).

With DENIS BONHEURE [36], MIGUEL has proved the *existence of an unbounded sequence of solutions for systems*

$$-\Delta u = |v|^{q-2}v + k(x), \quad -\Delta v = |u|^{p-2}u + h(x) \;\; in \;\; \Omega$$

$$u = v = 0 \;\; on \;\; \partial\Omega$$

when $p, q > 2$ *satisfy some growth restriction depending upon* N. Generic results are obtained under a weaker restriction.

An infinite sequence of solutions has also been obtained in [42] *for perturbed sublinear systems (*$1 < p, q < 2$*)*

$$-\Delta u = |v|^{p-2}u + g(x, v), \quad -\Delta v = |u|^{p-2}u + f(x, u) \;\; in \;\; \Omega$$

$$u = v = 0 \;\; on \;\; \partial\Omega$$

when $f(x, 0) = g(x, 0) = 0$ *satisfy suitable growth restrictions.*

9 Systems of elliptic equations

In a series of papers with JIANFU YANG [30], ANGELA PISTOIA [32, 34], SERGIO SOARES [33], and HUGOTAVARES [35] and alone [38], MIGUEL has considered the *existence and localization of peaks of lowest energy solutions of elliptic systems of the form*

$$-\varepsilon^2\Delta u + u = g(v), \quad -\varepsilon^2\Delta v + v = f(u) \;\; in \;\; \Omega$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \;\; or \;\; u = v = 0 \;\; on \;\; \partial\Omega$$

under superlinearity conditions for g *at* 0 *and* ∞*.*

Let us also mention a paper on Liouville type theorem for the system

$$-\Delta u = |v|^{q-2}v, \; -\Delta v = |u|^{p-2}u$$

in $\mathbb{R}^N$ [39].

In several recent works with DENIS BONHEURE and EDERSON MOREIRA DOS SANTOS [44, 46], MIGUEL has obtained *ground state and non-ground state solutions of strongly coupled elliptic systems on bounded, exterior domains or whole space.*

I finally mention some papers on elliptic systems related to population dynamics written with DJAIRO DE FIGUEIREDO [17], BENEDETTA NORIS [43], ANA RUTE DOMINGOS [45], and results on problems on the whole space or a ball for some elliptic equations with WANG ZHI-QIANG, WILLEM [23], PEDRO GIRÃO [27], WENMING ZOU [41].

10 Style

All of MIGUEL's papers have in common a lot a stylistic characteristics. First, **honesty**. MIGUEL's sources of inspiration are always very clearly and precisely indicated, and completed by a scrupulous description of the existing literature.

If you want to make the HISTORY OF A TOPICS that MIGUEL HAS TREATED, you will get it from his paper.

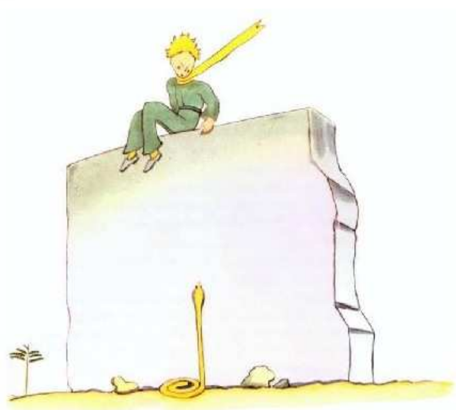
Most of MIGUEL's papers are **technically very involved** and you feel his strong taste for sharp inequalities, delicate estimates and elaborate constructions.

Not afraid of **algebraic topology**, he perfectly dominated and cleverly used all the topological tools which, as soon as you leaves the direct method of the calculus of variations, are necessary to develope critical point theory.

MIGUEL's papers also reveal his very good **mathematical taste**. There is no place in his work for expected generalizations or variants, for unmotivated applications. His papers answer questions that the majority of experts consider as important.

11 Conclusion

Miguel has fought the severe illness that took finally his life with the same courage and determination he had in all other aspects of his life. He fought for his family, for BEATRICE and his three daughters, he fought for his love and devotion for mathematics, cultivating them till the very end.



One year ago, his departure was a tragedy for his family and a severe loss for his many collaborators and friends all around the world. He had been for many years a devoted guide, he is now an example.

For a mathematician, the problem is in my opinion a shakespearien

to see or not to see.

There is no doubt that the Little Prince MIGUEL could even see inside a closed box.

We all are fortunate and richer, both on the human and the mathematical side, to have crossed his path.

I am sure that, on his little planet he has rejoined now, our Little Prince of mathematics continues to enjoy delightful mountain pass situations.

12 Publications of Miguel Ramos

I. PAPERS

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